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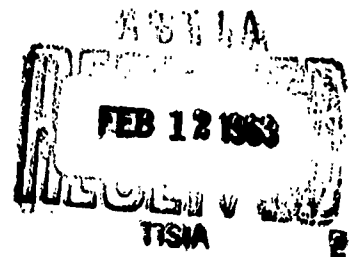
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The Exterior Problem for the Vector  
Wave Equation in an Inhomogeneous  
Anisotropic Medium



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William C. Hoffman

Mathematics Research

November 1962

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THE EXTERIOR PROBLEM FOR THE VECTOR WAVE EQUATION  
IN AN INHOMOGENEOUS ANISOTROPIC MEDIUM

by

William C. Hoffman

Mathematical Note No. 278

Mathematics Research Laboratory

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Abstract. The vector wave equation for an electromagnetic field outside a perfectly conducting sphere situated in an inhomogeneous anisotropic medium is reduced, by means of a dyadic Stratton-Chu formula, to an equivalent vector integral equation. The anisotropy is presumed to arise from a magnetic dipole at the center of the sphere, so that the kernel of the integral equation consists of the inner product of the appropriate conductivity tensor and the Green's dyadic (in a form due to C. T. Tai) for a sphere in free space. A discussion is given of the spherical vector wave functions involved, and various transformations are applied to render the vector integral equation more tractable. Finally the vector system is reduced to a single scalar integral equation apparently more suited to numerical solution through proper redefinition of the domain of integration.

1. Introduction. The vector wave equation of interest has the form

$$\text{curl curl } \underline{\underline{E}} = k_0^2 \underline{\underline{\kappa}} \cdot \underline{\underline{E}} \quad (1)$$

where  $\underline{\underline{E}}$  is the unknown field,  $k_0$  the free space wave number, and  $\underline{\underline{\kappa}}$  a dyadic, or second order tensor, that expresses the anisotropy of the medium. Such a differential equation arises naturally in several

contexts [1], but in the present instance interest is centered on the electromagnetic field. Eq. (1) comes about from Maxwell's equations in the following way. Maxwell's equations, in a form appropriate to an inhomogeneous anisotropic medium, read

$$\text{curl } \underline{\underline{E}} = \mu_0 \frac{\partial \underline{\underline{H}}}{\partial t}, \quad \text{curl } \underline{\underline{H}} = \underline{\underline{J}} + \epsilon_0 \frac{\partial \underline{\underline{E}}}{\partial t} = \frac{\partial \underline{\underline{D}}}{\partial t} \quad (2)$$

$$\text{div } \underline{\underline{D}}_{\text{eff}} = 0 \quad \text{div } \underline{\underline{B}} = 0 \quad (3)$$

where  $\underline{\underline{J}}$  is related to  $\underline{\underline{E}}$  by the conductivity tensor  $\underline{\underline{\sigma}}$ :

$$\underline{\underline{J}} = \underline{\underline{\sigma}} \cdot \underline{\underline{E}}, \quad (4)$$

and the (effective) electric induction  $\underline{\underline{D}}_{\text{eff}}$  is given by

$$\underline{\underline{D}}_{\text{eff}} = \epsilon_0 \underline{\underline{\kappa}} \cdot \underline{\underline{E}} \quad (5)$$

The effective dielectric tensor  $\underline{\underline{\kappa}}$  is related to the conductivity tensor by the equation

$$\underline{\underline{\kappa}} = \underline{\underline{I}} + \underline{\underline{\sigma}} / (i\omega\epsilon_0) \quad (6)$$

Assuming harmonic time dependence  $(e^{i\omega t})$  substitution of the second of eqs. (2) into the first then leads to the vector wave equation (1) upon taking account of (5).

If one expands the left hand side of (1) by the usual vector identity, a term in  $\text{grad div } \underline{\underline{E}}$  results. In an inhomogeneous anisotropic medium this term inextricably couples the field components. Thus not only the vectorial nature of the boundary conditions but also the

Since, for an arbitrary vector  $\underline{F}$  and dyadic  $\underline{G}$ , both in  $C_2$ ,

$$\nabla \cdot (\underline{F} \times \nabla \times \underline{G}) = \nabla \times \underline{F} \cdot \nabla \times \underline{G} - \underline{F} \cdot \nabla \times \nabla \times \underline{G}$$

and

$$\begin{aligned} \nabla \cdot (\underline{G} \times \nabla \times \underline{F}) &= \nabla \times \underline{F} \cdot \nabla \times \underline{G} - \nabla \times \nabla \times \underline{F} \cdot \underline{G} \\ &= \nabla \times \underline{F} \cdot \nabla \times \underline{G} - \underline{G}^T \cdot \nabla \times \nabla \times \underline{F}, \end{aligned}$$

where  $\underline{G}^T$  denotes the transpose of  $\underline{G}$ , the difference of the volume integrals

$$\int_V \nabla \cdot (\underline{F} \times \nabla \times \underline{G}) dV \quad \text{and} \quad \int_V \nabla \cdot (\underline{G} \times \nabla \times \underline{F}) dV$$

yields, in view of Gauss' theorem, the following vector Green's formula:

$$\int_V \{ \underline{F} \cdot \nabla \times \nabla \times \underline{G} - \underline{G}^T \cdot \nabla \times \nabla \times \underline{F} \} dV = \int_{\partial V} \underline{n} \cdot (\underline{G} \times \nabla \times \underline{F} - \underline{F} \times \nabla \times \underline{G}) d\Sigma \quad (7)$$

The terms in the surface integral may be expanded as follows:

$$\underline{n} \cdot \underline{G} \times \nabla \times \underline{F} = \nabla \times \underline{F} \cdot \underline{n} \times \underline{G} \quad \text{and} \quad \underline{n} \cdot \underline{F} \times \nabla \times \underline{G} = \underline{n} \times \underline{F} \cdot \nabla \times \underline{G}$$

and we finally obtain from (7) the desired generalization of the Stratton-Chu formula:

$$\begin{aligned} \int_V \{ \underline{F} \cdot \nabla \times \nabla \times \underline{G} - \nabla \times \nabla \times \underline{F} \cdot \underline{G} \} dV &= \int_V \{ \underline{F} \cdot \nabla \times \nabla \times \underline{G} - \underline{G}^T \cdot \nabla \times \nabla \times \underline{F} \} dV = \\ &= \int_{\partial V} \{ \nabla \times \underline{F} \cdot \underline{n} \times \underline{G} - \underline{n} \times \underline{F} \cdot \nabla \times \underline{G} \} d\Sigma. \end{aligned} \quad (8)$$

The order of the inner products in these integrands is essential.

Let us now apply formula (8) to the vector wave equation

$$\nabla \times \nabla \times \underline{E} = k_0^2 \underline{M} \cdot \underline{E} + \underline{J}(\underline{r}), \quad (9)$$



where  $\underline{J}(\underline{r})$  represents the current distribution of a finite source that may be present, the conduction current in the anisotropic conducting medium having already been included in  $\underline{\kappa} \cdot \underline{E}$ . We thus take  $\underline{F} = \underline{E}$  and  $\underline{G}$  to be the dyadic Green's function for a homogeneous isotropic medium:

$$\nabla \times \nabla \underline{G} = k_0^2 \underline{G} + \underline{I} \delta(\underline{P} - \underline{Q}), \quad (10)$$

but satisfying the same boundary and for radiation conditions as  $\underline{E}$ .

Thus at the surface of the perfectly conducting sphere,

$$\underline{n} \times \underline{E} = 0 \quad \text{and} \quad \underline{n} \times \underline{G} = 0 \quad (11)$$

Substitution of (9), (10), and (11) into (8) and taking account of Silver-Muller type radiation conditions for  $\underline{E}$  and  $\underline{G}$  then leads to the following equivalent integral equations for the electric field at point  $\underline{P}$  in an inhomogeneous anisotropic medium exterior to a perfectly conducting sphere:

$$\underline{E}(\underline{P}) = \underline{F}(\underline{P}) + k_0^2 \int_{V_a} \underline{E}(\underline{Q}) \cdot \underline{M}^T(\underline{Q}) \cdot \underline{G}(\underline{P}, \underline{Q}) d\underline{Q} \quad (12)$$

or

$$\underline{E}(\underline{P}) = \underline{F}(\underline{P}) + k_0^2 \int_{V_a} \underline{G}^T(\underline{P}, \underline{Q}) \cdot \underline{M}(\underline{Q}) \cdot \underline{E}(\underline{Q}) d\underline{Q} \quad (13)$$

where

$$\underline{F}(\underline{P}) = -i\omega\mu_0 \int_{V_a} \left\{ \begin{array}{ll} \underline{J}(\underline{Q}) \cdot \underline{G}(\underline{P}, \underline{Q}) d\underline{Q} & \text{for eq. (12)} \\ \underline{G}^T(\underline{P}, \underline{Q}) \cdot \underline{J}(\underline{Q}) d\underline{Q} & \text{for eq. (13)} \end{array} \right. \quad (14)$$

$V_a$  denotes that portion of 3-space exterior to the sphere of radius  $a$ , and

$$\underline{M} = \underline{\kappa} - \underline{I} = \underline{\sigma} / (i\omega\epsilon_0). \quad (15)$$

It is appropriate here to consider the forms that  $\underline{\underline{m}}$  (which, apart from multiplication constants, is essentially the conductivity tensor) can take in various magnetoplasmas. Each form is obtainable from the ponderomotive equation for a magneto-ionic medium:

$$\underline{\underline{m}} \frac{\partial \underline{v}}{\partial t} = q \{ \underline{E} + \underline{v} \times (\underline{B}_0 + \underline{b}) \} - \nu \underline{m} \underline{v}, \quad (16)$$

where  $q$  and  $m$  denote the charge and mass, respectively, of a charged particle moving with velocity  $\underline{v}$  in a medium with ambient magnetic field  $\underline{B}_0$  and average collision frequency  $\nu$ . The form of conductivity tensor is then determined in the usual way by assuming harmonic time dependence, expressing the vector product in (16) in the appropriate coordinate system, and introducing the polarization vector  $\underline{P}(\underline{r}) = (i\omega)^{-1} \underline{J}(\underline{r})$  and the notation of magneto-ionic theory [4]. For a uniform magnetic field parallel to the  $z$ -axis we have the familiar result

$$\underline{\underline{m}}_{\text{uniform}} = - \frac{X}{U^2 - Y^2} \begin{pmatrix} U & -iY & 0 \\ iY & U & 0 \\ 0 & 0 & (U^2 - Y^2)/U \end{pmatrix}. \quad (17)$$

For a magnetic vector  $\underline{B}_0$  lying entirely in the meridian plane in spherical coordinates the tensor  $\underline{\underline{m}}$  takes the form

$$\underline{\underline{m}}_{\text{sph}} = - \frac{X}{U^2 - Y^2} \begin{pmatrix} U^2 - Y_r^2 & -Y_r Y_\theta & iY_\theta \\ -Y_r Y_\theta & U^2 - Y_\theta^2 & -iY_r \\ -iY_\theta U & iY_r U & U^2 \end{pmatrix} \quad (18)$$

$$\text{where } Y_r = -eB_{or} / (m\omega) = 2Y \cos\theta / \sqrt{1 + 3 \cos^2\theta}$$

$$Y_\theta = -eB_{o\theta} / (m\omega) = Y \sin\theta / \sqrt{1 + 3 \cos^2\theta}$$

$$Y = \omega_H / \omega = 5.456 \times 10^6 \sqrt{1 + 3 \cos^2\theta} / [\omega(r/\alpha)^3]$$

For dipolar coordinates [5]  $\beta = \cos \theta / r^2$ ,  $\alpha = r / \sin^2 \theta$ ,  $\varphi = \omega$ ,  
with the ambient magnetic field being directed along a line of force  
 $\alpha = \text{const.}$  (i.e., parallel to  $\underline{l}_0$ ),  $\mathcal{M}$  takes essentially the same  
form as (17):

$$\mathcal{M}_{\text{dipolar}} = - \frac{X}{U^2 - Y^2} \begin{pmatrix} (U^2 - Y^2) / U & 0 & 0 \\ 0 & U & -iY \\ 0 & iY & U \end{pmatrix} \quad (19)$$

where  $Y = \omega_H / \omega$ ,

We note that in all the forms (17) - (19) the conductivity tensor  
consists of a factor  $X = \omega_p^2 / \omega^2$ , involving the electron density,  
multiplying a matrix which expresses only the effects of the ambient  
magnetic field and the collision frequency. Thus

$$\mathcal{M} = X(P) \mathcal{N}, \quad (20)$$

where the form of  $\mathcal{N}$  is immediately clear from the appropriate one of  
expressions (17) - (19).

### 3. The Green's Dyadic

#### A. Tai's Form of the Green's Dyadic

The Green's dyadic that appears in the integral equations (12) or (13) applies to the exterior problem for a perfectly conducting sphere situated in homogeneous isotropic space. The spectrum of the linear operator in (10) is therefore in general continuous, since the volume exterior to the sphere is infinite. To determine  $\mathbf{G}$  one can either employ an integral representation over a contour that guarantees outgoing waves at infinity or else adopt a series expansion in terms of vector spherical wave functions, each term of which consists of a generalized spherical harmonic in the "angular coordinates"  $\Theta$  and  $\varphi$ , multiplied by a discontinuous function of the radial coordinate. Since  $\mathbf{G}$  in the latter form is already available from the investigations of C. T. Tai [2], it will be adopted here. The transpose of Tai's Green's dyadic (this being the form appropriate to (13)), properly modified for  $e^{+i\omega t}$  time dependence, reads as follows:

$$\mathbf{G}_{\Sigma}^T(P, Q) = \sum_{n=1}^{\infty} c_n \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} \left\{ \begin{array}{l} \tilde{M}_{o\,mn}^r(P) \tilde{\Theta}_{o\,mn}^{(4)}(Q) + \tilde{N}_{o\,mn}^r(P) \tilde{H}_{o\,mn}^{(4)}(Q), \quad r_Q < r_P \\ \tilde{\Theta}_{o\,mn}(P) \tilde{M}_{o\,mn}^r(Q) + \tilde{H}_{o\,mn}(P) \tilde{N}_{o\,mn}^r(Q), \quad r_Q > r_P \end{array} \right. \quad (21a)$$

$$\tilde{\Theta}_{o\,mn}(P) \tilde{M}_{o\,mn}^r(Q) + \tilde{H}_{o\,mn}(P) \tilde{N}_{o\,mn}^r(Q), \quad r_Q > r_P \quad (21b)$$

where  $\epsilon_m$  is the Neumann symbol (1 for  $m = 0$ , 2 otherwise),

$$c_n = -ik_o \frac{2n+1}{n(n+1)}, \quad (22)$$

and  $\Theta_{\sim e_{mn}}^{(1,4)}$  and  $H_{\sim e_{mn}}^{(1,4)}$  are defined in terms of the spherical vector wave functions  $M_{\sim e_{mn}}^{(1,4)}$  and  $N_{\sim e_{mn}}^{(1,4)}$  as follows:

$$\Theta_{\sim e_{mn}}^{(1,4)}(P) = M_{\sim e_{mn}}^{(1)}(P) + R_n^b M_{\sim e_{mn}}^{(4)}(P) \quad (23)$$

$$H_{\sim e_{mn}}^{(1,4)}(P) = N_{\sim e_{mn}}^{(1)}(P) + R_n^e N_{\sim e_{mn}}^{(4)}(P) \quad (24)$$

The spherical vector wave functions themselves will be discussed in the next section. The reflection coefficients  $R_n^b$  and  $R_n^e$  are defined as follows:

$$R_n^b = - \frac{j_n(k_o a)}{h_n^{(2)}(k_o a)} \quad R_n^e = - \frac{[k_o a j_n(k_o a)]'}{[k_o a h_n^{(2)}(k_o a)]'}$$

In the sequel we shall often simply write  $r$  for the radial coordinate  $r_P$  of  $P$  and  $r'$  instead of  $r_Q$ .

We note that  $G_{<}^T(P, Q) = G_{>}(Q, P)$ ,  $G_{>}^T(P, Q) = G_{<}(Q, P)$ , simply interchanging antecedents and consequents in the dyadics.

#### B. Vector Spherical Wave Functions.

The scalar spherical wave functions have the form [2],[3,sec.7.11]

$$\psi_{\sim e_{mn}}(r, \Omega) = z_n(k_o r) P_n^m(\cos \theta) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\phi, \quad (25)$$

where  $z_n$  denotes a spherical Bessel function and  $\Omega$  is the angular variable  $(\theta, \varphi)$ . We note that

$$\nabla^2 \psi_{e_{mn}} + k_o^2 \psi_{e_{mn}} = 0$$

The spherical vector wave functions  $L_{e_{mn}}^r, M_{e_{mn}}^r, N_{e_{mn}}^r$  (the superscript  $r$  may on occasion be omitted) are then defined in terms of  $\psi_{e_{mn}}$  as follows:

$$L_{e_{mn}}^r = \nabla \psi_{e_{mn}} = \frac{dz_n(k_o r)}{dr} P_n^m(\cos \theta) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_m \varphi \underline{1}_r + r^{-1} z_n(k_o r) \frac{dP_n^m}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_m \varphi \underline{1}_\theta +$$

$$+ \frac{m}{r \sin \theta} z_n(k_o r) P_n^m(\cos \theta) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}_m \varphi \underline{1}_\varphi \quad (26)$$

$$M_{e_{mn}}^r = \nabla \times (\psi_{e_{mn}} \underline{r}) = \underline{L}_{e_{mn}} \times \underline{r} =$$

$$= + \frac{m}{\sin \theta} z_n(k_o r) P_n^m(\cos \theta) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}_m \varphi \underline{1}_\theta - z_n(k_o r) \frac{dP_n^m}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_m \varphi \underline{1}_\varphi \quad (27)$$

$$N_{e_{mn}}^r = k_o^{-1} \nabla \times \nabla \times (\psi_{e_{mn}} \underline{r}) = n(n+1) \frac{z_n(k_o r)}{k_o r} P_n^m(\cos \theta) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_m \varphi \underline{1}_r +$$

$$+ \frac{[k_o r z_n(k_o r)]'}{k_o r} \frac{dP_n^m}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_m \varphi \underline{1}_\theta +$$

$$+ \frac{m}{\sin \theta} \frac{[k_o r z_n(k_o r)]'}{k_o r} P_n^m(\cos \theta) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}_m \varphi \underline{1}_\varphi \quad (28)$$

$\tilde{M}_{o\ mn}^r$  and  $\tilde{N}_{o\ mn}^r$  have a reciprocal relation to each other:

$$\tilde{M}_{o\ mn}^r = k_o^{-1} \nabla \times \tilde{N}_{o\ mn}^r, \quad \tilde{N}_{o\ mn}^r = k_o^{-1} \nabla \times \tilde{M}_{o\ mn}^r$$

Then we may write (21) as

$$G^T(P, Q) = \sum_{n=1}^{\infty} c_n \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} \left\{ \begin{aligned} & \nabla_P \times (\psi_{e\ mn}^{(4)}) \nabla_Q \times [(\psi_{e\ mn}^{(1)}(Q) + R_n^b \psi_{e\ mn}^{(4)}(Q))r] + \\ & + k_o^{-2} \nabla_P \times \nabla_P \times (\psi_{e\ mn}^{(4)}) \nabla_Q \times \nabla_Q \times [(\psi_{e\ mn}^{(1)}(Q) + R_n^e \psi_{e\ mn}^{(4)}(Q))r], \quad r' < r \\ & \nabla_P \times [(\psi_{e\ mn}^{(1)}(P) + R_n^b \psi_{e\ mn}^{(4)}(P))r] \nabla_Q \times (\psi_{e\ mn}^{(4)}(Q)r') + \\ & + k_o^{-2} \nabla_P \times \nabla_P \times [(\psi_{e\ mn}^{(1)}(P) + R_n^e \psi_{e\ mn}^{(4)}(P))r] \nabla_Q \times \nabla_Q \times (\psi_{e\ mn}^{(4)}(Q)r'), \end{aligned} \right. \quad r' > r$$

This form of  $G^T(P, Q)$  is helpful in expanding the dyadic into components.

The expansion

$$P_n(\cos \gamma) = \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{m(n+m)!} P_n^m(\cos \theta_P) P_n^m(\cos \theta_Q) \cos m(\varphi_P - \varphi_Q) \quad (29)$$

will play an important role in what follows. It will, however, be more convenient for our purposes to rewrite (29) in terms of the even and odd spherical harmonics

$$Y_{e\ mn}(\Omega) = P_n^m(\cos \theta) \begin{Bmatrix} \cos m \varphi \\ \sin m \varphi \end{Bmatrix}. \quad (30)$$

Thus

$$\psi_{e_{mn}}(P) = z_n(k_o r) Y_{e_{mn}}(\Omega), \quad (31)$$

which leads to an even and odd resolution of (29):

$$\begin{aligned} P_n(\cos \gamma) &= P_{e,n}(\cos \gamma) + P_{o,n}(\cos \gamma) = \\ &= \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} [Y_{e,mn}(\Omega) Y_{e,mn}(\Omega') + Y_{o,mn}(\Omega) Y_{o,mn}(\Omega')] \end{aligned} \quad (32)$$

The spherical vector wave functions, (30) and (31), become

$$\underline{L}_{e_{mn}}(P) = \frac{\partial z_n(k_o r)}{\partial r} Y_{e_{mn}}(\Omega) \underline{l}_r + r^{-1} z_n(k_o r) \frac{\partial Y_{e_{mn}}}{\partial \theta} \underline{l}_\theta + \frac{z_n(k_o r)}{r \sin \theta} \frac{\partial Y_{e_{mn}}}{\partial \varphi} \underline{l}_\varphi \quad (33)$$

$$\underline{M}_{e_{mn}}(P) = z_n(k_o r) \left\{ \frac{1}{\sin \theta} \frac{\partial Y_{e_{mn}}}{\partial \varphi} \underline{l}_\theta - \frac{\partial Y_{e_{mn}}}{\partial \theta} \underline{l}_\varphi \right\} \quad (34)$$

$$\begin{aligned} \underline{N}_{e_{mn}}(P) &= k_o r \psi_{e_{mn}} \underline{l}_r + 2k_o^{-1} \nabla \psi_{e_{mn}} + k_o^{-1} (\underline{r} \cdot \nabla) \nabla \psi_{e_{mn}} \\ &= (k_o r)^{-1} \{ n(n+1) z_n(k_o r) Y_{e_{mn}}(\Omega) \underline{l}_r + [k_o r z_n(k_o r)]' \left[ \frac{\partial Y_{e_{mn}}}{\partial \theta} \underline{l}_\theta + \frac{1}{\sin \theta} \frac{\partial Y_{e_{mn}}}{\partial \varphi} \underline{l}_\varphi \right] \} \end{aligned} \quad (35)$$

We are now in a position to expand the dyadic products in  $\underline{G}^T(P, Q)$ .

Thus

$$\begin{aligned} \underline{M}_{e_{mn}}(P) \underline{M}_{e_{mn}}(Q) &= z_n(k_o r) z_n(k_o r') \left\{ \underline{l}_\theta \frac{\frac{\partial Y_{e_{mn}}}{\partial \varphi}}{\sin \theta \sin \theta'} \underline{l}_{\theta'} - \right. \\ &\quad - \underline{l}_\theta \frac{1}{\sin \theta} \frac{\partial Y_{e_{mn}}}{\partial \varphi} \frac{\partial Y_{e_{mn}}}{\partial \theta'} \underline{l}_{\varphi'} - \underline{l}_\varphi \frac{1}{\sin \theta'} \frac{\partial Y_{e_{mn}}}{\partial \theta} \frac{\partial Y_{e_{mn}}}{\partial \varphi'} \underline{l}_{\theta'} + \\ &\quad \left. + \underline{l}_\varphi \frac{\partial Y_{e_{mn}}}{\partial \theta} \frac{\partial Y_{e_{mn}}}{\partial \theta'} \underline{l}_{\varphi'} \right\} \end{aligned} \quad (36)$$



and, defining

$$f_n(k_o r) = [k_o z_n(k_o r)]', \quad g_n(k_o r') = [k_o r' z_n(k_o r')], \quad (37)$$

$$\begin{aligned} N_{o mn}^{(P)} N_{o mn}^{(Q)} = & (k_o^2 r r')^{-1} \{ 1_r [n(n+1)]^2 z_n(k_o r) z_n(k_o r') Y_{o mn}(\Omega) Y_{o mn}(\Omega') 1_{r'} + \\ & + n(n+1) z_n(k_o r) g_n(k_o r') [1_r Y_{o mn}(\Omega) \frac{\partial Y_{o mn}}{\partial \Theta'} 1_{\Theta'} + \\ & + 1_r \frac{1}{\sin \Theta'} Y_{o mn}(\Omega) \frac{\partial Y_{o mn}}{\partial \Phi'} 1_{\Phi'}] + \\ & + n(n+1) f_n(k_o r) z_n(k_o r') [-\frac{\partial Y_{o mn}}{\partial \Theta} 1_{\Theta} + \\ & + \frac{1}{\sin \Theta} \frac{\partial Y_{o mn}}{\partial \Phi} 1_{\Phi}] Y_{o mn}(\Omega') 1_{r'} + \\ & + f_n(k_o r) g_n(k_o r') [1_{\Theta} \frac{\partial Y_{o mn}}{\partial \Theta} \frac{\partial Y_{o mn}}{\partial \Theta'} 1_{\Theta'} + \\ & + 1_{\Theta} \frac{1}{\sin \Theta'} \frac{\partial Y_{o mn}}{\partial \Theta} \frac{\partial Y_{o mn}}{\partial \Phi'} 1_{\Phi'} + \\ & + 1_{\Phi} \frac{1}{\sin \Theta} \frac{\partial Y_{o mn}}{\partial \Phi} \frac{\partial Y_{o mn}}{\partial \Theta'} 1_{\Theta'} + 1_{\Phi} \frac{\partial Y_{o mn}}{\partial \Phi} \frac{\partial Y_{o mn}}{\partial \Phi'} \frac{1}{\sin \Theta'} 1_{\Phi'}] \} \end{aligned} \quad (38)$$

The summations on  $m$  in (21) can then be expressed as

$$\sum_{m=0}^n e_m \frac{(n-m)!}{(n+m)!} M_{\circ mn}(P) M_{\circ mn}(Q) = z_n(k_o r) Z_n(k_o r') \left\{ \frac{\partial^2 P_{\circ mn}(\cos \gamma)}{\partial \varphi \partial \varphi'} \frac{1}{\sin \theta \sin \theta'} \right. \\ \left. - \frac{1}{\sin \theta} \frac{\partial^2 P_{\circ n}}{\partial \varphi \partial \theta'} \frac{1}{\varphi'} - \frac{1}{\sin \theta'} \frac{\partial^2 P_{\circ n}}{\partial \theta \partial \varphi'} \frac{1}{\varphi} + \frac{1}{\sin \theta \sin \theta'} \frac{\partial^2 P_{\circ n}}{\partial \theta \partial \theta'} \frac{1}{\varphi \varphi'} \right\} \quad (39)$$

and

$$\sum_{m=0}^n e_m \frac{(n-m)!}{(n+m)!} N_{\circ mn}(P) N_{\circ mn}(Q) = (k_o^2 r r')^{-1} \left\{ \frac{1}{r} [n(n+1)]^2 z_n(k_o r) Z_n(k_o r') P_{\circ n}(\cos \gamma) \frac{1}{r'} + \right. \\ \left. + n(n+1) z_n(k_o r) Z_n(k_o r') \frac{1}{r} \frac{\partial P_{\circ n}}{\partial \theta'} \frac{1}{\varphi'} + \frac{1}{r} \frac{1}{\sin \theta'} \frac{\partial P_{\circ n}}{\partial \varphi'} \frac{1}{\varphi \varphi'} + \right. \\ \left. + n(n+1) z_n(k_o r) Z_n(k_o r') \frac{1}{r} \frac{\partial P_{\circ n}}{\partial \theta} \frac{1}{r'} + \frac{1}{r} \frac{1}{\sin \theta} \frac{\partial P_{\circ n}}{\partial \varphi} \frac{1}{r' \varphi'} + \right. \\ \left. + z_n(k_o r) Z_n(k_o r') \frac{1}{r} \frac{\partial^2 P_{\circ n}}{\partial \theta \partial \theta'} \frac{1}{\varphi \varphi'} + \frac{1}{r} \frac{1}{\sin \theta} \frac{\partial^2 P_{\circ n}}{\partial \theta \partial \varphi'} \frac{1}{\varphi \varphi'} + \right. \\ \left. + \frac{1}{r} \frac{1}{\sin \theta} \frac{\partial^2 P_{\circ n}}{\partial \varphi \partial \theta'} \frac{1}{\varphi \varphi'} + \frac{1}{r} \frac{1}{\sin \theta \sin \theta'} \frac{\partial^2 P_{\circ n}}{\partial \varphi \partial \varphi'} \frac{1}{\varphi \varphi'} \right\} \quad (40)$$

We can thus write (21) as a dyadic wherein the summation on  $m$  no longer appears explicitly:

$$\begin{aligned}
 \mathbf{Q}_{<}^T(P, Q) = & \sum_{n=1}^{\infty} c_n \{ [j_n(k_o r_{<}) + R_{nn}^e h(k_o r_{<})] h(k_o r_{>}) k_o r_{>} \}^{-1} [n(n+1)]^2 P_{e_n}(\cos \gamma) l_{r_{<}} + \\
 & + l_{\theta} \left[ \frac{Q_1(r_{<}, r_{>})}{\sin \theta \sin \theta'} \frac{\partial^2 P_{e_n}}{\partial \varphi \partial \varphi'} + Q_2(r_{<}, r_{>}) \frac{\partial^2 P_{e_n}}{\partial \theta \partial \theta'} \right] l_{\theta'} - \\
 & - l_{\theta} \left[ \frac{1}{\sin \theta} Q_1(r_{<}, r_{>}) \frac{\partial^2 P_{e_n}}{\partial \varphi \partial \theta'} - \frac{1}{\sin \theta'} Q_2(r_{<}, r_{>}) \frac{\partial^2 P_{e_n}}{\partial \theta \partial \varphi'} \right] l_{\varphi'} - \\
 & - l_{\varphi} \left[ \frac{1}{\sin \theta'} Q_1(r_{<}, r_{>}) \frac{\partial^2 P_{e_n}}{\partial \theta \partial \varphi'} - \frac{1}{\sin \theta} Q_2(r_{<}, r_{>}) \frac{\partial^2 P_{e_n}}{\partial \varphi \partial \theta'} \right] l_{\theta'} + \\
 & + l_{\varphi} \left[ Q_1(r_{<}, r_{>}) \frac{\partial^2 P_{e_n}}{\partial \theta \partial \theta'} + \frac{Q_2(r_{<}, r_{>})}{\sin \theta \sin \theta'} \frac{\partial^2 P_{e_n}}{\partial \varphi \partial \varphi'} \right] l_{\varphi'} + \quad (41a) \\
 & + n(n+1)(k_o^2 r_{<} r_{>})^{-1} \left\{ \begin{aligned} & [j_n(k_o r_{<}) + R_{nn}^e h(k_o r_{<})] j_n^{(1)}(k_o r_{>}) l_{r_{<}} \frac{\partial P_{e_n}}{\partial \varphi'} l_{\theta'} + l_{r_{<}} \frac{\partial P_{e_n}}{\partial \varphi'} l_{\varphi'} + \\ & + [j_n^{(1)}(k_o r_{<}) + R_{nn}^e j_n^{(1)}(k_o r_{<})] h(k_o r_{>}) l_{\theta} \frac{\partial P_{e_n}}{\partial \theta} l_{r_{>}} + l_{\varphi} \frac{\partial P_{e_n}}{\partial \theta} l_{r_{>}} \frac{1}{\sin \theta} l_{r_{>}} \frac{\partial P_{e_n}}{\partial \varphi'} l_{\varphi'} + \\ & + h(k_o r_{>}) j_n^{(1)}(k_o r_{<}) + R_{nn}^e j_n^{(1)}(k_o r_{<}) l_{r_{>}} \frac{\partial P_{e_n}}{\partial \theta'} l_{\theta'} + l_{r_{>}} \frac{\partial P_{e_n}}{\partial \theta'} l_{\varphi'} \frac{1}{\sin \theta'} l_{\varphi'} + \\ & + j_n^{(1)}(k_o r_{>}) [j_n(k_o r_{<}) + R_{nn}^e h(k_o r_{<})] l_{\theta} \frac{\partial P_{e_n}}{\partial \theta} l_{r_{<}} + l_{\varphi} \frac{1}{\sin \theta} \frac{\partial P_{e_n}}{\partial \varphi} l_{r_{<}} \frac{\partial P_{e_n}}{\partial \varphi'} l_{\varphi'} \end{aligned} \right. \\
 & \quad \quad \quad r' > r_{<} \\
 & \quad \quad \quad r' < r_{>}
 \end{aligned}
 \tag{41b}$$

where

$$Q_1(r_<, r_>) = [j_n(k_0 r_<) + R_{nn}^b(k_0 r_<)] h_n(k_0 r_>) \quad (42)$$

$$Q_2(r_<, r_>) = (k_0^2 r_< r_>)^{-1} [j_n^{(1)}(k_0 r_<) + R_{nn}^e(k_0 r_<)] j_n^{(4)}(k_0 r_>). \quad (43)$$

#### 4. Transformations of the Vector Integral Equation

##### A. Reduction to Symmetric Form

It will now be assumed that the function  $X(P)$  (i.e., the plasma frequency) exhibits radial symmetry, and further that  $X(r)$  vanishes for  $r > r_0$ . Then the integral equation (13) takes the form

$$\begin{aligned} \underline{E}(P) = \underline{F}(P) + k_0^2 \{ & \int_a^r dr' r'^2 X(r') \int_{\Omega'} \underline{G}_{<}^T(P, Q) \cdot \underline{N}(Q) \cdot \underline{E}(Q) d\Omega' + \\ & + \int_r^{r_0} dr' r'^2 X(r') \int_{\Omega'} \underline{G}_{>}^T(P, Q) \cdot \underline{N}(Q) \cdot \underline{E}(Q) d\Omega' \} \end{aligned} \quad (44)$$

Evaluation of  $\underline{F}(P)$  according to (14) for a horizontal (i.e., parallel to  $\underline{1}_\theta$ ) current element of moment  $p$  at the point  $(b, \theta_1, \varphi_1)$  above the sphere:

$$\underline{J}_H(r, \Omega) = i\omega p \frac{\delta(r-b)\delta(\theta-\theta_1)\delta(\varphi-\varphi_1)}{r^2 \sin \theta} \underline{1}_\theta,$$

yields, for horizontal polarization,

$$\begin{aligned}
 F_H(P) = \frac{k_o^2}{\epsilon_o} \sum_{n=1}^{\infty} c_n \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} & \left\{ \begin{aligned} & [M_{\sim e_{mn}}^r(1)(b, \theta_1, \varphi_1) + R_{n \sim e_{mn}}^b M_{\sim e_{mn}}^r(4)(b, \theta_1, \varphi_1)] M_{\sim e_{mn}}^r(P) + \\ & + [N_{\sim e_{mn}}^r(1)(b, \theta_1, \varphi_1) + R_{n \sim e_{mn}}^e N_{\sim e_{mn}}^r(4)(b, \theta_1, \varphi_1)] N_{\sim e_{mn}}^r(P), \end{aligned} \right. \\
 & r > b \\
 & (45) \\
 & \left\{ \begin{aligned} & M_{\sim e_{mn}}^r(4)(b, \theta_1, \varphi_1) [M_{\sim e_{mn}}^r(P) + R_n^b M_{\sim e_{mn}}^r(P)] + \\ & + N_{\sim e_{mn}}^r(4)(b, \theta_1, \varphi_1) [N_{\sim e_{mn}}^r(P) + R_n^e N_{\sim e_{mn}}^r(P)], \end{aligned} \right. \\
 & r < b
 \end{aligned}$$

For vertical polarization, a vertical current element at  $(b, \theta_1, \varphi_1)$ ,

$$J_V(r, \Omega) = i\omega p \frac{\delta(r-b)\delta(\theta-\theta_1)\delta(\varphi-\varphi_1)}{r^2 \sin \theta} \underline{1}_r,$$

leads in the same way to

$$F_V(P) = \frac{k_o^2}{\epsilon_o} \sum_{n=1}^{\infty} c_n \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} \left\{ \begin{aligned} & [N_{r_{\sim e_{mn}}}^r(1)(b, \theta_1, \varphi_1) + R_{n r_{\sim e_{mn}}}^e N_{r_{\sim e_{mn}}}^r(4)(b, \theta_1, \varphi_1)] N_{\sim e_{mn}}^r(P), r > b \\ & [N_{r_{\sim e_{mn}}}^r(4)(b, \theta_1, \varphi_1) [N_{\sim e_{mn}}^r(P) + R_{n r_{\sim e_{mn}}}^e N_{\sim e_{mn}}^r(P)]], r < b \end{aligned} \right. \quad (46)$$

An attempt to expand  $\tilde{E}(P)$  in the usual way in a series of spherical vector wave functions

$$\tilde{E}(P) = \sum_{q=1}^{\infty} \sum_{p=0}^q \{ e_{pq}^{(11)}(r) \tilde{M}_{o,pq}^{(1)}(P) + e_{pq}^{(14)}(r) \tilde{M}_{o,pq}^{(4)}(P) + e_{pq}^{(21)}(r) \tilde{N}_{o,pq}^{(1)}(P) + e_{pq}^{(24)}(r) \tilde{N}_{o,pq}^{(4)}(P) \}$$

is foredoomed to failure, since the iterated summation causes the expansion coefficients in each row of the infinite system of algebraic equations resulting from the substitution of (47) into (44) to be different. Thus unless some device (like a generalization of the Watson transformation) is available that permits one to discard, for each  $q$ , all but the first two coefficients, it will not be possible to solve for the expansion coefficients in (47) in the usual way.

It may be possible to carry out such an expansion in terms of the spherical Bessel functions and the Legendre polynomials (32) in  $\gamma$ . However, even then it would be more convenient to have (44) in symmetric form to assure the applicability of the usual theorems on eigenvalues, eigenfunctions and developability for symmetric kernels [6],[7]. Thus we define the new (definite) unknown function and source function

$$\tilde{\mathcal{E}}(P) = \mathcal{N}^{\frac{1}{2}}(P) \cdot \tilde{E}(P) \quad \text{and} \quad \tilde{\mathcal{F}}(P) = \mathcal{N}^{\frac{1}{2}}(P) \cdot \tilde{F}(P) \quad (48 \text{ a,b})$$

and the symmetric kernel

$$\mathcal{K}_{\geq}(P, Q) = \mathcal{N}^{\frac{1}{2}}(P) \cdot \mathcal{G}_{\geq}^T(P, Q) \cdot \mathcal{N}^{\frac{1}{2}}(Q) \quad (49)$$

Then the integral equation (44) can be written in symmetric form as

$$\begin{aligned} \tilde{\ell}(P) - k_0^2 \left\{ \int_a^r dr' r'^2 X(r') \int_{\Omega'} K_<(P, Q) \cdot \tilde{\ell}(Q) d\Omega' + \right. \\ \left. + \int_r^{r_0} dr' r'^2 X(r') \int_{\Omega'} K_>(P, Q) \cdot \tilde{\ell}(Q) d\Omega' \right\} = \tilde{\mathcal{J}}(P) \end{aligned} \quad (50)$$

We next must give a specific form for (49), which involves first of all the determination of  $\mathcal{M}^{\frac{1}{2}}$ . For this purpose we employ the method of matrix Lagrange polynomials [8] and restrict ourselves to the spherical coordinate form of  $\mathcal{M}$  (eq. (18)) with  $U = 1$  (i.e., the collision frequency  $\nu = 0$ ). The first step is to solve the characteristic equation  $|\mathcal{M} - \Lambda \mathbf{I}| = 0$  for the 3 eigenvalues of the matrix  $\mathcal{M}$ . The characteristic equation is the cubic

$$\Lambda^3 + \frac{3 - Y^2}{1 - Y^2} \Lambda^2 + \frac{3}{1 - Y^2} \Lambda + \frac{1}{1 - Y^2} = 0,$$

or

$$(\Lambda + 1)^3 - Y^2 \Lambda^2 (\Lambda + 1) = 0,$$

whence it follows that one eigenvalue is

$$\Lambda_1 = -1. \quad (51)$$

The reduced quadratic  $(\Lambda + 1)^2 - Y^2 \Lambda^2 = 0$  then yields the other two eigenvalues

$$\Lambda_{2,3} = \Lambda_{\pm} = -(1 \pm Y)^{-1} \quad (52)$$

Then [9,p.232]  $\mathcal{M}^{\frac{1}{2}}$  can be represented in terms of matrix Lagrange polynomials  $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$  as

$$\mathcal{M}^{\frac{1}{2}} = \Lambda_1^{\frac{1}{2}} \mathcal{L}_1 + \Lambda_2^{\frac{1}{2}} \mathcal{L}_2 + \Lambda_3^{\frac{1}{2}} \mathcal{L}_3, \quad (53)$$

where

$$\mathcal{L}_1 = \frac{\mathcal{P}_2 \mathcal{P}_3}{\lambda_{12} \lambda_{13}}, \quad \mathcal{L}_2 = \frac{\mathcal{P}_1 \mathcal{P}_3}{\lambda_{21} \lambda_{23}}, \quad \mathcal{L}_3 = \frac{\mathcal{P}_1 \mathcal{P}_2}{\lambda_{31} \lambda_{32}} \quad (54)$$

with

$$\mathcal{P}_k = \mathcal{M} - \Lambda_k \mathbf{I}, \quad \lambda_{ik} = \Lambda_i - \Lambda_k, \quad (i, k = 1, 2, 3)$$

Thus

$$\lambda_{12} = -Y(1+Y)^{-1} = -\lambda_{21}$$

$$\lambda_{13} = Y(1-Y)^{-1} = -\lambda_{31}$$

$$\lambda_{23} = 2Y(1-Y^2)^{-1} = -\lambda_{23}$$

and

$$\mathcal{P}_1 = -\frac{1}{1-Y^2} \begin{pmatrix} Y_\theta^2 & -Y_r Y_\theta & iY_\theta \\ -Y_r Y_\theta & Y_r^2 & -iY_r \\ -iY_\theta & iY_r & Y^2 \end{pmatrix}$$

$$\mathcal{P}_2 = \frac{1}{1-Y^2} \begin{pmatrix} Y-Y_r^2 & -Y_r Y_\theta & iY_\theta \\ -Y_r Y_\theta & Y-Y_\theta^2 & -iY_r \\ -iY_\theta & iY_r & Y \end{pmatrix}$$



$$\rho_3 = \frac{1}{1 - Y^2} \begin{pmatrix} Y + Y_r^2 & Y_r Y_\theta & -iY_\theta \\ Y_r Y_\theta & Y + Y_\theta^2 & iY_r \\ iY_\theta & -iY_r & Y \end{pmatrix}$$

Then the matrix interpolation polynomials are, according to (54),

$$\mathcal{L}_1 = \begin{pmatrix} Y_r^2 & Y_r Y_\theta & 0 \\ Y_r Y_\theta & Y_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (55)$$

$$\mathcal{L}_2 = \frac{1}{2} \begin{pmatrix} Y_\theta^2 & -Y_r Y_\theta & -iY_\theta \\ -Y_r Y_\theta & Y_r^2 & iY_r \\ iY_\theta & -iY_r & 1 \end{pmatrix} = \mathcal{L}_3^* \quad (56)$$

where  $Y_r = Y_r/Y$ ,  $Y_\theta = Y_\theta/Y$ . (57)

Thus, according to (53),

$$\eta^{\frac{1}{2}} = i \left\{ \mathcal{L}_1 + \frac{\mathcal{L}_2}{\sqrt{1+Y}} + \frac{\mathcal{L}_2^*}{\sqrt{1-Y}} \right\}$$

or

$$n^{\frac{1}{2}} = i \begin{pmatrix} 1 - y_0^2(1 - \frac{1}{2}y_+) & y_r y_0(1 - \frac{1}{2}y_+) & -\frac{1}{2}i y_0 y_- \\ y_r y_0(1 - \frac{1}{2}y_+) & 1 - y_r^2(1 - \frac{1}{2}y_+) & \frac{1}{2}i y_r y_- \\ \frac{1}{2}i y_0 y_- & -\frac{1}{2}i y_r y_- & \frac{1}{2}y_+ \end{pmatrix} \quad (58)$$

where

$$y_{\pm} = \frac{1}{\sqrt{1+Y}} \pm \frac{1}{\sqrt{1-Y}} \quad (59)$$

For the sake of brevity we shall shorten the notation for the elements of

 $n^{\frac{1}{2}}$  as follows:

$$n^{\frac{1}{2}} = i \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \eta_4 & \eta_5 \\ \eta_3^* & \eta_5^* & \eta_6 \end{pmatrix}$$

Then, according to (49), the elements of  $-K_{\geq}(P, Q)$  are given by

$$\begin{aligned} K_{11}(P, Q) &= \eta_1(P)[G_{rr}^T, \eta_1(Q) + G_{r\theta}^T, \eta_2 + G_{r\varphi}^T, \eta_3^*] + \eta_2(P)[G_{\theta r}^T, \eta_1 + G_{\theta\theta}^T, \eta_2 + G_{\theta\varphi}^T, \eta_3^*] + \\ &\quad + \eta_3(P)[G_{\varphi r}^T, \eta_1 + G_{\varphi\theta}^T, \eta_2 + G_{\varphi\varphi}^T, \eta_3^*] \\ K_{12}(P, Q) &= \eta_1(P)[G_{rr}^T, \eta_2(Q) + G_{r\theta}^T, \eta_4 + G_{r\varphi}^T, \eta_5^*] + \eta_2(P)[G_{\theta r}^T, \eta_2 + G_{\theta\theta}^T, \eta_4 + G_{\theta\varphi}^T, \eta_5^*] + \\ &\quad + \eta_3(P)[G_{\varphi r}^T, \eta_2 + G_{\varphi\theta}^T, \eta_4 + G_{\varphi\varphi}^T, \eta_5^*] \end{aligned}$$

$$K_{13}(P, Q) = \eta_1(P)[G_{rr}^T, \eta_3(Q) + G_{r\theta}^T \eta_5 + G_{r\varphi}^T \eta_6] + \eta_2(P)[G_{\theta r}^T, \eta_3 + G_{\theta\theta}^T \eta_5 + G_{\theta\varphi}^T \eta_6] +$$

$$+ \eta_3(P)[G_{\varphi r}^T, \eta_3 + G_{\varphi\theta}^T \eta_5 + G_{\varphi\varphi}^T \eta_6]$$

$$K_{21}(P, Q) = \eta_2(P)[G_{rr}^T, \eta_1(Q) + G_{r\theta}^T \eta_2 + G_{r\varphi}^T \eta_3^*] + \eta_4(P)[G_{\theta r}^T, \eta_1 + G_{\theta\theta}^T \eta_2 + G_{\theta\varphi}^T \eta_3^*] +$$

$$+ \eta_5(P)[G_{\varphi r}^T, \eta_1 + G_{\varphi\theta}^T \eta_2 + G_{\varphi\varphi}^T \eta_3^*] \quad (61)$$

$$K_{22}(P, Q) = \eta_2(P)[G_{rr}^T, \eta_2(Q) + G_{r\theta}^T \eta_4 + G_{r\varphi}^T \eta_5^*] + \eta_4(P)[G_{\theta r}^T, \eta_2 + G_{\theta\theta}^T \eta_4 + G_{\theta\varphi}^T \eta_5^*] +$$

$$+ \eta_5(P)[G_{\varphi r}^T, \eta_2 + G_{\varphi\theta}^T \eta_4 + G_{\varphi\varphi}^T \eta_5^*]$$

$$K_{23}(P, Q) = \eta_2(P)[G_{rr}^T, \eta_3(Q) + G_{r\theta}^T \eta_5 + G_{r\varphi}^T \eta_6] + \eta_4(P)[G_{\theta r}^T, \eta_3 + G_{\theta\theta}^T \eta_5 + G_{\theta\varphi}^T \eta_6] +$$

$$+ \eta_5(P)[G_{\varphi r}^T, \eta_3 + G_{\varphi\theta}^T \eta_5 + G_{\varphi\varphi}^T \eta_6]$$

$$K_{31}(P, Q) = \eta_3^*(P)[G_{rr}^T, \eta_1(Q) + G_{r\theta}^T \eta_2 + G_{r\varphi}^T \eta_3^*] + \eta_5^*(P)[G_{\theta r}^T, \eta_1 + G_{\theta\theta}^T \eta_2 + G_{\theta\varphi}^T \eta_3^*] +$$

$$+ \eta_6(P)[G_{\varphi r}^T, \eta_1 + G_{\varphi\theta}^T \eta_2 + G_{\varphi\varphi}^T \eta_3^*]$$

$$K_{32}(P, Q) = \eta_3^*(P)[G_{rr}^T, \eta_2(Q) + G_{r\theta}^T \eta_4 + G_{r\varphi}^T \eta_5^*] + \eta_5^*(P)[G_{\theta r}^T, \eta_2 + G_{\theta\theta}^T \eta_4 + G_{\theta\varphi}^T \eta_5^*] +$$

$$+ \eta_6(P)[G_{\varphi r}^T, \eta_2 + G_{\varphi\theta}^T \eta_4 + G_{\varphi\varphi}^T \eta_5^*]$$

$$K_{33}(P, Q) = \eta_3^*(P)[G_{rr}^T, \eta_3(Q) + G_{r\theta}^T \eta_5 + G_{r\varphi}^T \eta_6] + \eta_5^*(P)[G_{\theta r}^T, \eta_3 + G_{\theta\theta}^T \eta_5 + G_{\theta\varphi}^T \eta_6] +$$

$$+ \eta_6(P)[G_{\varphi r}^T, \eta_3 + G_{\varphi\theta}^T \eta_5 + G_{\varphi\varphi}^T \eta_6]$$

The complexity of the elements (61) casts doubt upon the appropriateness of an attempt to determine the expansion coefficients in the usual way. However, a numerical approach appears possible, and we now direct our attention towards a form of (50) more suited for numerical solution.

#### B. Reduction to a Single Scalar Integral Equation

In the case of one independent variable a well-defined method exists for reducing a system of integral equations to a single scalar integral equation [7, sec.17], [9,p.1]. The range of integration must be finite, and the vector integral equation is reduced to scalar form by simply redefining the kernel and other functions involved over an extended range of integration. It is possible to apply a similar method to (50), and we now proceed to its derivation.

Let the independent variables  $P$  and  $Q$  in (50) be respectively

$$P = (r_p, \Omega) \qquad Q = (r_Q, \Omega') \quad , \qquad (62)$$

and define new independent variables

$$P' = (\rho, \Omega), \qquad Q' = (\rho', \Omega') \qquad (63)$$

over the spherical shell-like regions characterized by the following inequalities:

$$r + (j - 1)(r_0 - a) < \rho < a + j(r_0 - a) \qquad (j, k = 1, 2, 3) \qquad (64)$$

$$r' + (k - 1)(r_0 - a) < \rho' < a + k(r_0 - a)$$

Now the vector equation (50), when written in the form of a system, is equivalent to the following:

$$\begin{aligned} \mathcal{E}_j(P) - k_0^2 \sum_{k=1}^3 \left\{ \int_a^r dr' r'^2 X(r') \int_{\Omega'} K_{jk}(P, Q) \mathcal{E}_k(Q) d\Omega' + \int_r^{r_0} dr' r'^2 X(r') \int_{\Omega'} K_{jk}(P, Q) \mathcal{E}_k(Q) d\Omega' \right\} = \\ = \mathcal{F}_j(P) \quad (j = 1, 2, 3) \end{aligned} \quad (65)$$

If we now define the new functions

$$\xi(P') = \mathcal{E}_j(P) \quad \mathcal{F}(P') = \mathcal{F}_j(P) \quad (66)$$

and the new kernel

$$\mathcal{K}(P', Q') = X(r') K_{jk}(P, Q) = X(r') \begin{cases} K_{jk}(P, Q), & r > r' \\ K_{jk}(P, Q), & r < r' \end{cases} \quad (67)$$

then (65) can be written as the single scalar equation

$$\xi(\rho, \Omega) - k_0^2 \int_a^{3r_0 - 2a} \int_{\Omega'} \mathcal{K}(\rho, \rho'; \Omega, \Omega') \xi(\rho', \Omega') \rho'^2 d\rho' d\Omega' = \mathcal{F}(\rho, \Omega). \quad (68)$$

Although (68) is still entirely equivalent to (65) it appears to possess certain advantages in that we have to do with a scalar rather than a vector integral equation and that the "bookkeeping" for machine computation is spelled out explicitly in (66) and (67).

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